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ALTERNATIVE FOR THE ENCOUNTER-EVASION DIFFERENTIAL GAME WITH CONSTRAINTS ON THE MOMENTA OF THE PLAYERS' CONTROLS

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We examine a differential game in which the players can control the system's motion with the aid of generalized impulses. Similar problems were investigated in [1 – 5]. In this paper we describe position procedures of control with a guide, within the framework of which we establish alternative conditions for the solvability of encounter and evasion problems. The fundamental constructions used in the paper are similar to the extremal construction proposed in [6 – 8] for differential games with geometric constraints on the players' controls.

1. We examine a differential game in which the motion of a conflict-controlled system is described by the equation

$$x^{\cdot} = f(t, x) + B(t) U^{\cdot} + C(t) V^{\cdot} + x_0 \delta(t - t_0) \tag{1.1}$$

Here x is the system's n-dimensional phase vector, the function $f(t, x)$ is continuous in all arguments and satisfies a Lipschitz condition in x, U and V are vector-valued controls of the first and second players, of dimensions l_1 and l_2, $B(t)$ and $C(t)$ are continuous matrix-valued functions of appropriate dimensions, $\delta(t - t_0)$ is a delta-function, x_0 is the system's initial state at the instant $t = t_0$. Equation (1.1) is examined on the semiaxis $t \geqslant t_0$ and is to be understood in the sense of the theory of distributions [9]. As the admissible controls $U[t]$ and $V[t]$ we choose functions identically equal to zero for $t < t_0$ and right-continuous. We assume that the variations of the admissible controls satisfy the relations

$$\int_{t_0}^{\vartheta} \| dU[t] \| \leqslant \mu_0, \qquad \int_{t_0}^{\vartheta} \| dV[t] \| \leqslant \nu_0 \tag{1.2}$$

Here μ_0 and ν_0 are numbers specifying the control resources of the first and second players, $[t_0, \vartheta]$ is the time interval on which the game is examined; the variations of the controls are given by the relations

$$\int_{t_0}^{\vartheta} \| dU[t] \| = \| U[t_0] \| + \sup \sum_{i=1}^{k} \| U[\tau_i] - U[\tau_{i-1}] \| \tag{1.3}$$

$$\int_{t_0}^{\vartheta} \| dV[t] \| = \| V[t_0] \| + \sup \sum_{i=1}^{k} \| V[\tau_i] - V[\tau_{i-1}] \|$$

where the upper bound is taken over all possible finite partitionings of the interval $[t_0, \vartheta]$ $(\tau_0 = t_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_k = \vartheta)$, the symbol $\| F \|$ denotes the Euclidean norm of vector F. For each pair of admissible controls $U[t]$ and $V[t]$ a solution of system (1.1) exists and is unique, viz. a function $x[t]$ of bounded variation, right-continuous, identically equal to zero for $t < t_0$, and satisfying system (1.1) in the sense of the theory of distributions (cf. [9]).

The differential game being investigated here is composed of the encounter problem and the evasion problem. The encounter problem, facing the first payer, consists in ensuring that the point $\{t, x[t]\}$ reaches a given set M^* at some specified instant $t = \vartheta$ and is guaranteeing here the fulfillment of the phase constraint $\{t, x[t]\} \in N^*$ up to the contact of point $\{t, x[t]\}$ with set M^*. The evasion problem, facing the second player, consists in guaranteeing that either the point $\{t, x[t]\}$ evades contact with set M^* up to the instant $t = \vartheta$ or the phase constraint $\{t, x[t]\} \in N^*$ is violated before the point $\{t, x[t]\}$ reaches set M^*.

We are required to find the solutions of these problems in the class of position control methods using information only on the game positions realized. Here, by a game position realized at instant t we mean the vector

$$p[t] = \{t, x[t-0], \mu[t-0], \nu[t-0]\}$$

where the symbol $s[t-0]$ denotes the limit from the left of the function $s[\tau]$ at the point $\tau = t$; the quantities $\mu[t]$ and $\nu[t]$ estimate the control resources available to the first and second players at instant t; these quantities are specified by the relations

$$\mu[t] = \mu_0 - \int_{t_0}^{t} \|dU[\tau]\|, \quad \nu[t] = \nu_0 - \int_{t_0}^{t} \|dV[\tau]\|$$

We examine the case when the sets M^* and N^* are closed and are cylindrical in the directions of the axes of those coordinates x_j for which the corresponding components $x_j[t]$ of motions $x[t]$ can be affected under the impulse actions of the controls. We prove that in this case, for any initial game position $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ and for any number $\vartheta \geqslant t_0$, either the first player's problem of encounter with set M^* at instant $t = \vartheta$ or the second player's problem of evasion up to instant $t = \vartheta$ is always solvable. In the proof we use an extremal construction analogous to the constructions in [6-8] and modified with due regard to the impulse character of the controls.

2. Let us introduce the elements of the extremal construction used in solving the encounter problem. We consider the following auxiliary system:

$$x^{\cdot} = f(t, x) + B(t) U_*^{\cdot} + C(t) V_*^{\cdot} + x_* \delta(t - t_0) \tag{2.1}$$

where the vector $p_* = \{t_*, x_*, \mu_*, \nu_*\}$ plays the role of the initial position. The concepts of admissible controls and their resources for this system are obtained from the corresponding concepts for system (1.1) replacing $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ by $p_* = \{t_*, x_*, \mu_*, \nu_*\}$. Here we assume that

$$V_*^{\cdot}(t) = v_* \delta(t - t_*) \tag{2.2}$$

where v_* is some l_2-dimensional vector satisfying the condition $\| v_* \| \leqslant \nu_*$.

Definition 2.1. The collection of points $p^* = \{t^*, x^*, \mu^*, \nu^*\}$ of the form

$$x^* = x_* + \int_{t_*}^{t_*} f(t, x[t])\, dt + \int_{t_*-0}^{t^*} B(t)\, dU_*(t) + C(t_*) v_*$$

$$0 \leqslant \mu^* \leqslant \mu_* - \int_{t_*}^{t^*} \|dU_*(t)\|, \quad \nu^* = \nu_* - \|v_*\|$$

where $U_*(t)$ are arbitrary admissible controls of system (2.1) and $x[t]$ are the corresponding solutions of system (2.1), is called the attainability region $G(t^*, p_*, V_*)$ of system (2.1), constructed for the instant $t = t^*$ from the initial position p_* under a fixed control $V_*(t)$ of (2.2).

Here

$$\int_{t_*-0}^{t_*} B(t)\,dU_*(t) = \lim_{\Delta\to 0} \int_{t_*-\Delta}^{t^*} B(t)\,dU_*(t), \quad \Delta > 0$$

the integrals in the right-hand side of this relation are to be understood in the sense of Stieltjes. We note that the set $G(t^*, p_*, V_*)$ turns out to be closed. We introduce into consideration the sets

$$M = [p = \{t, x, \mu, \nu\} : \{t, x\} \in M^*, \ \mu \geqslant 0, \ \nu \geqslant 0]$$
$$N = [p = \{t, x, \mu, \nu\} : \{t, x\} \in N^*, \ \mu \geqslant 0, \ \nu \geqslant 0]$$

Let D be some set in the space of vectors $p = \{t, x, \mu, \nu\}$ and let the symbol D_τ denote the section of this set by the hyperplane $t = \tau$.

Definition 2.2. Let $W^{(u)}$ be some set in the space of vectors $p = \{t, x, \mu, \nu\}$. We say that this set is a u-stable bridge in the encounter problem if

$$W^{(u)} \subset N, \quad W_\vartheta^{(u)} \subset M$$

and if one of the following two conditions (τ is some point of the interval $[t_*, t^*]$)

$$G(t^*, p_*, V_*) \cap W^{(u)} \neq \phi, \ G(\tau, p_*, V_*) \cap M \neq \phi$$

is valid for any point $p_* = \{t_*, x_*, \mu_*, \nu_*\} \in W^{(u)}$, for any instant $t^* \in [t_*, \vartheta]$ and for every control V_* of form (2.2).

We note that in this definition we do not exclude the case $W_\vartheta^{(u)} = \phi$. It can be shown that the closure of every u-stable bridge is, as before, a u-stable bridge. In what follows we examine only closed bridges $W^{(u)}$.

Let us describe the first player's position control method which yields the solution of the encounter problem. The construction proposed here is based on the guide-control scheme, similar to the constructions in [7, 8]. Let $W^{(u)}$ be a u-stable bridge and let $W_{t_0}^{(u)} \neq \phi$. By $p_0^* = \{t_0, x_0^*, \mu_0^*, \nu_0^*\}$ we denote the point of set $W_{t_0}^{(u)}$ nearest to the point $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$. We select some covering of interval $[t_0, \vartheta]$ by equal intervals

$$[\tau_i, \tau_{i+1}), \ i = 0, 1, \ldots, m-1; \ \tau_0 = t_0, \ \tau_m = \vartheta, \ \tau_{i+1} = \tau_i + \Delta$$

We assume

$$r_1(t_0) = \begin{cases} \mu_0 - \mu_0^*, & \mu_0 > \mu_0^* \\ 0, & \mu_0 \leqslant \mu_0^* \end{cases}, \quad r_2(t_0) = \begin{cases} \nu_0^* - \nu_0, & \nu_0^* > \nu_0 \\ 0, & \nu_0^* \leqslant \nu_0 \end{cases}$$

We determine the vectors u_0 and v_0^* from the condition

$$\|x_0 - x_0^* + B(t_0)u_0 - C(t_0)v_0^*\| = \min_{u,v} \|x_0 - x_0^* + B(t_0)u - C(t_0)v\| \quad (2.3)$$
$$\text{for} \quad \|u\| \leqslant r_1(t_0), \ \|v\| \leqslant r_2(t_0)$$

We determine the first player's control in system (1.1) for $t_0 \leqslant t < t_0 + \Delta$ by the relation

$$U_\Delta^{\cdot}[t] = u_0\delta(t - t_0)$$

This control, in pair with some control of the second player, realizes a motion of system (1.1) for $t_0 \leqslant t < \tau_1$. To construct the control $U_\Delta[t]$ for $\tau_i \leqslant t < \tau_{i+1}$ we determine the point

$$p_i^* = p^*[\tau_i] = \{\tau_i, x^*[\tau_i], \mu^*[\tau_i], \nu^*[\tau_i]\} \in G(\tau_i, p_{i-1}^*, V_{i-1}^*) \cap W^{(u)}$$

assuming here that the intersection of the sets is nonempty, while the control V_{i-1}^* is given by the relation

$$V_{i-1}^{*\cdot}(t) = v_{i-1}^* \delta(t - \tau_{i-1}) \tag{2.4}$$

We determine the quantities

$$r_1(\tau_i) = \min\{\mu^*[\tau_{i-1}] - \mu^*[\tau_i], \quad \mu[\tau_i - 0]\}$$
$$r_2(\tau_i) = \min\{\nu[\tau_{i-1} - 0] - \nu[\tau_i - 0], \ \nu^*[\tau_i]\}$$

Vectors u_i and v_i^* are found from the condition

$$\|x[\tau_i - 0] - x^*[\tau_i] + B(\tau_i)u_i - C(\tau_i)v_i^*\| = \min_{u,v}\|x[\tau_i - 0] - x^*[\tau_i] + B(\tau_i)u - C(\tau_i)v\| \tag{2.5}$$
$$\text{for} \quad \|u\| \leqslant r_1(\tau_i), \ \|v\| \leqslant r_2(\tau_i)$$

and we assume

$$U_\Delta^\cdot[t] = u_i\delta(t - \tau_i), \quad \tau_i \leqslant t < \tau_{i+1} \tag{2.6}$$

This control, in pair with some addmissible control $V[t]$ of the second player, realizes the motion of the system (1.1) for $\tau_i \leqslant t < \tau_{i+1}$. If the condition

$$G(\tau_i, p_{i-1}^*, V_{i-1}^*) \cap W^{(u)} \neq \phi \tag{2.7}$$

is fulfilled for $i = 1, 2, \ldots, m-1$, then the procedure described above specifies the control $U_\Delta[t]$ of (2.5), (2.6) up to the last instant $t = \vartheta$.

Let us now consider the case when condition (2.7) is satisfied only for $i = 1, 2, \ldots, j < m-1$. Then, we determine the control $U_\Delta[t]$ by the above-mentioned method up to the instant $t = \tau_j$. Next, we determine the control $U_\Delta[t]$ for $\tau_j \leqslant t \leqslant \vartheta$ from relations (2.5), (2.6), assuming that for $i > j$ the points p_i^* are chosen arbitrarily from the sets $G(\tau_i, p_{i-1}^*, V_{i-1}^*)$. Note that by construction $p_j^* \in W^{(u)}$, therefore, from the condition

$$G(\tau_{j+1}, p_j^*, V_j^*) \cap W^{(u)} = \phi$$

follows (see Definition 2.2) the existence of an instant $\tau^* \in [\tau_j, \tau_{j+1}]$, for which the relation

$$G(\tau^*, p_j^*, V_j^*) \cap M \neq \phi$$

is valid, i.e. in this case we can say that the point $\{\tau_j, x^*(\tau_j)\}$ is located not far from set M^*. Note also that the choice of controls $U_\Delta[t]$ and $V^*(t)$ from conditions (2.4) – (2.6) ensures the mutual tracking of the motions of the original system (1.1) and of the guide whose positions at the instant $t = \tau_i$ are denoted here by $p^*[\tau_i]$.

The motion of system (1.1), generated by the control $U_\Delta[t]$ of (2.5), (2.6), is denoted by the symbol $x_\Delta[t, p_0, V[\cdot], W^{(u)}]$, where $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ is the initial position, Δ is the step size of the recurrence procedure, $V[t]$ $(t \geqslant t_0)$ is the second player's admissible control realized in system (1.1) in pair with the control $U_\Delta[t]$; $W^{(u)}$ is a u-stable bridge for which the guide-control procedure being examined was constructed. These motions are called approximated. Together with the appro ximated motions we introduce the motions determined by passing to the limit in the sequence of approximated motions.

Definition 2.3. The summable function $x(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ for which there exists the sequence of approximated motions

$$x_k[t] = x_{\Delta_k}[t,\ p_0^{(k)},\ V^{(k)}[\cdot],\ W^{(u)}],\quad t_0 \leqslant t \leqslant \vartheta,\quad k = 1, 2, \ldots$$

converging to it in the metric of space $L[t_0, \vartheta]$ and satisfying the conditions

$$\Delta_k \to 0,\quad p_0^{(k)} = \{t_0,\ x_0^{(k)},\ \mu_0^{(k)},\ \nu_0^{(k)}\} \to p_0 = \{t_0, x_0, \mu_0, \nu_0\}$$
for $k \to \infty$

is called a motion $x(t, p_0, W^{(u)})$ $(t_0 \leqslant t \leqslant \vartheta)$ of system (1.1), generated by the first player's guide-control procedure.

We note that the existence of the motions $x(t, p_0, W^{(u)})$ can be established by using Helly's theorem.

The following assertion is valid.

Theorem 2.1. If the initial position $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ is contained by the u-stable bridge $W^{(u)}$, then for any motion $x(t, p_0, W^{(u)})$ the point $\{t, x(t, p_0, W^{(u)})\}$ reaches set M^* not later than at the instant $t = \vartheta$ remaining in the set N^* up to contact with set M^*.

Let us indicate the salient features of the proof of this theorem. Let us consider the approximated motion $x_\Delta[t, p_0^{(\Delta)}, V[\cdot], W^{(u)}]$ $(t_0 \leqslant t \leqslant \vartheta)$. Simultaneously with the realization of this motion we form, by the method defined above, the guide's motion, i.e. we select the points $p_i^* = p^*[\tau_i]$ moving along the bridge $W^{(u)}$ and reach set M at some instant $t = \tau^*$. The controls $U_\Delta[t]$ and $V^*(t)$ are chosen from conditions (2.4)–(2.6) so as to ensure the mutual tracking of the motions of the original system and the motions of the guide. Thus, the proof of this theorem is reduced to estimating the distance between the motions of the original system (1.1) and the guide. We note that it is convenient to carry out the required estimation of the distance between the motions $x_\Delta[t]$ and $x_\Delta^*[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ in the metric of space $L[t_0, \vartheta]$; it can be established that this distance satisfies the relation

$$\rho(x_\Delta[\cdot],\ x_\Delta^*[\cdot]) \to 0 \quad \text{as} \quad \Delta \to 0 \tag{2.8}$$

uniformly for all the motions being examined under the condition that $\| p_0^{(\Delta)} - p_0^{*(\Delta)} \| \to 0$ as $\Delta \to 0$.

From estimate (2.8) we conclude that uniform convergence obtains for all those components of the vector-valued functions $x_\Delta[t]$ and $x_\Delta^*[t]$, which are not affected under the impulse actions of the controls on systems (1.1), (2.1). Therefore, the situation formulated in Theorem 2.1 acquires the following meaning: if $p_0 \in W^{(u)}$, then for arbitrary number $\varepsilon > 0$ there exists $\Delta(\varepsilon) > 0$ such that when $\Delta < \Delta(\varepsilon)$, the point $\{t, x_\Delta[t]\}$ falls into the ε-neighborhood of set M^* for every approximated motion $x_\Delta[t] = x_\Delta[t, p_0, V[\cdot], W^{(u)}]$, remaining in the ε-neighborhood of set N^* up to contact with the ε-neighborhood of set M^*.

3. Let us consider the evasion problem. When investigating this problem we take advantage of the construction introduced above to solve the encounter problem.

Definition 3.1. The collection of points $p^* = \{t^*, x^*, \mu^*, \nu^*\}$ of the form

$$x^* = x_* + \int_{t_*}^{t^*} f(t, x[t])\,dt + \int_{t_*-0}^{t^*} C(t)\,dV_*(t) + B(t_*)u_*$$

$$0 \leqslant \nu^* \leqslant \nu_* - \int_{t_*}^{t^*} \|dV_*(t)\|, \quad \mu^* = \mu_* - \|u_*\|$$

(where $V_*(t)$ are arbitrary admissible controls of system (2.1) and $x[t]$ are the corresponding solutions of system (2.1)), is called the attainability region $G(t^*, p_*, U_*)$ of system (2.1), constructed for the instant $t = t^* \geqslant t_*$ from the initial position $p_* = \{t_*, x_*, \mu_*, \nu_*\}$ under a fixed control $U_*(t)$ of the form

$$U_*(t) = u_*\delta(t - t_*), \quad \|u_*\| \leqslant \mu_* \tag{3.1}$$

Let G and H be sets defined by the relations

$$G = [p = \{t, x, \mu, \nu\}: \{t, x\} \in G^*, \ \mu \geqslant 0, \ \nu \geqslant 0] \tag{3.2}$$
$$H = [p = \{t, x, \mu, \nu\}: \{t, x\} \in H^*, \ \mu \geqslant 0, \ \nu \geqslant 0]$$

where G^* and H^* are certain closed sets satisfying the conditions

$$G^* \cap M^* = \varnothing, \ H^* \cap N^* = \varnothing \tag{3.3}$$

Definition 3.2. Let $W^{(v)}$ be some set in the space of vectors $p = \{t, x, \mu, \nu\}$. We say that this set is a v-stable bridge in the evasion problem if sets G and H of form (3.2), (3.3) exist such that the condition

$$W^{(v)} \subset G$$

is satisfied, and if one of the following two relations (τ is some point of interval $[t_*, t^*]$):

$$G(t^*, p_*, U_*) \cap W^{(v)} \neq \varnothing, \ G(\tau, p_*, U_*) \cap H \neq \varnothing$$

for any point $p_* = \{t_*, x_*, \mu_*, \nu_*\} \in W^{(v)}$, for any instant $t^* \in [t_*, \vartheta]$, and for every control U_* of form (3.1) is valid.

For the v-stable bridge $W^{(v)}$ we introduce the second player's guide-control procedure which differs from the first player's control method described above only in the replacement of sets M and N by the sets H and G, respectively, and in the interchange of the roles of the first and second players. The motions of system (1.1), generated by these control methods are denoted by the symbol $x_\Delta[t, p_0, U[\cdot], W^{(v)}]$, where $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ is the initial position; Δ is the step size of the recurrence procedure; $U[t]$ $(t \geqslant t_0)$ is the first player's admissible control realized in system (1.1) in pair with the control $V_\Delta[t]$ prescribed by the guide-control procedure; $W^{(v)}$ is the v-stable bridge for which this second player's control procedure was constructed. These motions are called approximated. Next, we introduce motions defined as the limits of certain sequences of approximated motions.

Definition 3.3. A summable function $x(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ for which there exists the sequence of approximated motions

$$x_k[t] = x_{\Delta_k}[t, p_0^{(k)}, U^{(k)}[\cdot], W^{(v)}], \quad t_0 \leqslant t \leqslant \vartheta, \quad k = 1, 2, \ldots$$

converging to it in the metric of space $L[t_0, \vartheta]$ and satisfying the conditions

$$\Delta_k \to 0, \ p_0^{(k)} = \{t_0, x_0^{(k)}, \mu_0^{(k)}, \nu_0^{(k)}\} \to p_0 = \{t_0, x_0, \mu_0, \nu_0\} \quad \text{at} \quad k \to \infty$$

is called a motion $x(t, p_0, W^{(v)})$ $(t_0 \leqslant t \leqslant \vartheta)$ of system (1.1), generated by the second player's guide-control procedure.

The following statement is valid.

Theorem 3.1. If the initial position $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ is contained by the v-stable bridge $W^{(v)}$, then for any motion $x(t, p_0, W^{(v)})$ the point $\{t, x(t, p_0, W^{(v)})\}$ remains in set G^* either for all $t \in [t_0, \vartheta]$ or up to the instant τ^* when this point first reaches set H^*.

Sets G^* and H^* satisfy relations (3.2), (3.3); therefore, this theorem establishes that the evasion conditions are satisfied for the motions $x(t, p_0, W^{(v)})$. We note that the meaning of Theorem 3.1, as well as Theorem 2.1, is revealed when considering the approximated motions (see the analogous note at the end of Sect. 2).

4. Let us indicate certain properties of the construction introduced, which we use below in the proof of the alternative for the encounter-evasion differential game.

Lemma 4.1. The union of a finite number of v-stable bridges $W_i^{(v)}$ is a v-stable bridge.

The validity of this statement is established by a direct verification of the properties of a v-stable bridge for the set

$$W^{(v)} = \bigcup_{i=1}^{k} W_i^{(v)}$$

Note that the sets G and H corresponding to the bridge $W^{(v)}$ are obtained as the unions

$$G = \bigcup_{i=1}^{k} G_i, \quad H = \bigcup_{i=1}^{k} H_i$$

where G_i and H_i are the sets corresponding to the bridges $W_i^{(v)}$.

Lemma 4.2. For any point $p_* = \{t_*, x_*, \mu_*, \nu_*\}$ belonging to a v-stable bridge $W^{(v)}$ there exist a neighborhood of this point

$$S_\varepsilon(p_*) = [p = \{t_*, x, \mu, \nu\}: \|x - x_*\| \leqslant \varepsilon, \ |\mu - \mu_*| \leqslant \varepsilon$$
$$|\nu - \nu_*| \leqslant \varepsilon, \ \mu \geqslant 0, \ \nu \geqslant 0]$$

and a v-stable bridge $W_*^{(v)}$ such that $S_\varepsilon(p_*) \subset W_*^{(v)}$.

In the proof of this lemma we use the property of semicontinuous dependency of the motions $x(t, p_*, W^{(v)})$ $(t_* \leqslant t \leqslant \vartheta)$ on the initial position $p_* = \{t_*, x_*, \mu_*, \nu_*\}$. This property consists in the following: if $p_*^{(k)} = \{t_*, x_*^{(k)}, \mu_*^{(k)}, \nu_*^{(k)}\} \to p_* = \{t_*, x_*, \mu_*, \nu_*\}$, $x(\cdot, p_*^{(k)}, W^{(v)}) \to x_*(\cdot)$ as $k \to \infty$ (here the convergence is in the metric of $L[t_*, \vartheta]$), then the function $x_*(t)$ $(t_* \leqslant t \leqslant \vartheta)$ is one of the motions $x(t, p_*, W^{(v)})$.

Lemma 4.3. Let W_0 be the union of all possible v-stable bridges, W° be the complement of set W_0. Then set W° is a u-stable bridge in the encounter problem.

Let us mention the highlights of the proof of this lemma. At first we show the fulfillment of the conditions

$$W^\circ \subset N, \qquad W_\vartheta^\circ \subset M \tag{4.1}$$

Every set H of form (3.2), (3.3) is a v-stable bridge. Therefore, the union of all such sets which coincide with the complement of set N is contained in the set W_0, i.e. $W^\circ \subset N$. On the other hand, the set G_ϑ also is a v-stable bridge for any set G of form (3.2), (3.3). Therefore, the union of all such sets, coinciding with the complement of set M_ϑ, is contained in W_0; consequently, $W_\vartheta^\circ \subset M_\vartheta$. Thus, conditions (4.1) are proved.

It remains to show the fulfillment of the condition of u-stability of set W°. We perform this by contradiction. Let there exist a point $p_* = \{t_*, x_*, \mu_*, \nu_*\} \in W^\circ$, an instant $t^* \in [t_*, \vartheta]$, and a control V_* of form (2.2) such that the relations

$$\begin{array}{ll} G(t^*, p_*, V_*) \cap W^\circ = \varnothing & \\ G(t, p_*, V_*) \cap M = \varnothing & \text{for } t_* \leqslant t \leqslant t^* \end{array} \tag{4.2}$$

are satisfied. From (4.2) we obtain the imbedding $G(t^*, p_*, V_*) \subset W_0$, i.e. for every point $p \in G(t^*, p_*, V_*)$ we can find a v-stable bridge $W^{(v)}$ containing it. By virtue of Lemma 4.2, for arbitrary point $p \in G(t^*, p_*, V_*)$, there exists a neighborhood $S(p)$ imbedded in some v-stable bridge $W_*^{(v)}$, and containing this point. Since the attainability region $G(t^*, p_*, V_*)$ is closed, then from an infinite covering of it by such neighborhoods $S(p)$ we can select a finite covering $S(p_i)$ $(i = 1, 2, \ldots, k)$. The union

$$W_*^{(v)} = \bigcup_{i=1}^{k} W_{*i}^{(v)} \tag{4.3}$$

of bridges $W_{*i}^{(v)}$ containing the neighborhoods $S(p_i)$ introduced, is once again a v-stable bridge, and the relation

$$W_*^{(v)} \supset G(t^*, p_*, V_*) \tag{4.4}$$

is valid for it.

Let us now consider the set

$$W_{**}^{(v)} = [p = \{t, x, \mu, \nu\}:$$

$$p \in G(t, p_*, V_*) \quad \text{for} \quad t_* \leqslant t < t^*; \quad p \in W_*^{(v)} \quad \text{for} \quad t^* \leqslant t \leqslant \vartheta]$$

From relations (4.4) – (4.6) we deduce that the set $W_{**}^{(v)}$ is a v-stable bridge, it is not difficult to see that $p_* \in W_{**}^{(v)}$; consequently, $p_* \in W_0$ and $p_* \notin W^0$. The contradiction obtained proves Lemma 4.3.

Now it is easy to establish the validity of the following alternative.

Theorem 4.1. For arbitrary initial position $p_0 = \{t_0, x_0, \mu_0, \nu_0\}$ and the number $\vartheta \geqslant t_0$, either the encounter problem or the evasion problem is solvable.

In fact, by virtue of Lemma 4.3 the space of positions $p = \{t, x, \mu, \nu\}$ $(\mu \geqslant 0, \nu \geqslant 0)$ splits into two parts: $W^{(u)} = W^\circ$ and $W_0 = \bigcup W^{(v)}$. If $p_0 \in W^\circ$, then by virtue of Theorem 2.1 the encounter problem is solvable; its solution provides a guide-control procedure defined for the bridge $W^\circ = W^{(u)}$. If, however, $p_0 \notin W^\circ$, then $p_0 \in W_0$, i.e. the point p_0 belongs to some v-stable bridge $W^{(v)}$, and then the evasion problem is solvable, and the solution of this problem provides a guide-control procedure defined for the v-stable bridge $W^{(v)}$. We note that when realizing the control procedures described here, as the stable bridges we can use sets determined by program absorption operations (in regular cases), by recurrence procedures, and by direct methods (see [1, 3, 5], for example).

5. As an illustration of the proposed control method we present the solution of the following simple pursuit problem. Let the motions of the pursuing and pursued objects be described by the one-type equations

$$\begin{array}{ll} y_1^{\cdot} = y_3 + y_{01}\delta(t - t_0), & y_2^{\cdot} = y_4 + y_{02}\delta(t - t_0) \\ y_3^{\cdot} = -\alpha y_3 + U_1^{\cdot} + y_{03}\delta(t - t_0), & y_4^{\cdot} = -\alpha y_4 + U_2^{\cdot} + y_{04}\delta(t - t_0) \\ z_1^{\cdot} = z_3 + z_{01}\delta(t - t_0), & z_2^{\cdot} = z_4 + z_{02}\delta(t - t_0) \\ z_3^{\cdot} = -\alpha z_3 + V_1^{\cdot} + z_{03}\delta(t - t_0), & z_4^{\cdot} = -\alpha z_4 + V_2^{\cdot} + z_{04}\delta(t - t_0) \end{array}$$

with the fulfillment of conditions (1.2) $(\mu_0 > \nu_0)$. Using information only on the game position realized, we are required to construct a control method U which ensures the contact of objects y and z at a certain finite instant. Here, by the contact of objects y and z we mean the coincidence of the first two coordinates of the phase vectors y and z, , i.e. the fulfillment of the equalities $y_1[t] = z_1[t]$ and $y_2[t] = z_2[t]$. It can be verified that the set

$$W^{(u)} = [p = \{t,\ y,\ z,\ \mu,\ \nu\} : y - z = y_0 - z_0,\ \ \mu - \nu = \mu_0 - \nu_0 \ \text{ for } \ t = t_0;\ y - z = x^\circ(t),\ \mu - \nu = \xi^\circ(t) \ \text{ for } \ t_0 < t \leqslant \vartheta^\circ]$$

is a u-stable bridge in the problem being examined. Here $x^\circ(t)$, $\xi^\circ(t)$ is the solution of the problem of the time-optimal transition of the system

$$x^{\cdot}_1 = x_3 + (y_{01} - z_{01})\,\delta(t - t_0)$$
$$x^{\cdot}_2 = x_4 + (y_{02} - z_{01})\,\delta(t - t_0)$$
$$x^{\cdot}_3 = -\alpha x_3 + W^{\cdot}_1 + (y_{03} - z_{03})\,\delta(t - t_0)$$
$$x^{\cdot}_4 = -\alpha x_4 + W^{\cdot}_2 + (y_{04} - z_{04})\,\delta(t - t_0)$$
$$0 \leqslant \xi(t) = \mu_0 - \nu_0 - \int_{t_0}^{\vartheta} \|dW(t)\|$$

to the state $x_1(\vartheta^\circ) = x_2(\vartheta^\circ) = 0$. The guide-control procedure for the bridge mentioned was implemented on a computer. The following initial data and parameters were selected:

$$y_{01} = 0, \quad y_{02} = y_{03} = y_{04} = 1$$
$$z_{01} = z_{02} = z_{03} = 0, \quad z_{04} = -1$$
$$\alpha = 1,\ \mu_0 = 4.74593,\ \nu_0 = 1.16395$$
$$t_0 = 0, \quad \vartheta^\circ = 1, \quad \Delta = 0.001$$

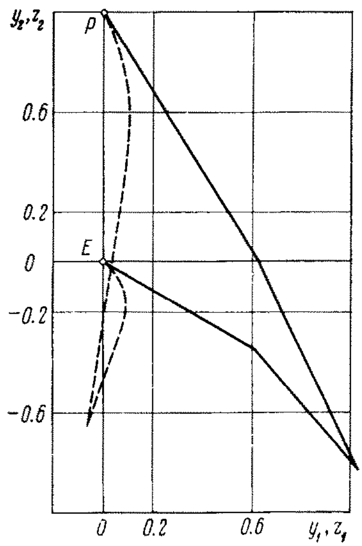

Fig. 1

The solid lines in Fig. 1 depict the trajectories of points y and z in the case

$$V^{\cdot}[t] = \{{}^1\!/_2\nu_0\delta(t),\ -{}^1\!/_2\nu_0\delta(t - {}^1\!/_2)\} = \{V_1^{\cdot}[t],\ V_2^{\cdot}[t]\}$$

The dashed lines picture the pursuit process for the case

$$V^{\cdot}[t] = \{-4\nu_0,\ 0\}, \quad 0 \leqslant t \leqslant {}^1\!/_4; \quad V^{\cdot}[t] \equiv 0, \quad t > {}^1\!/_4$$

Under the parameters selected, contact is realized at the instant $t = \vartheta^\circ = 1$. With respect to the first two coordinates the mismatch between the corresponding motions of the original system and the guide does not exceed an amout $\varepsilon = 0.003$.

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NEW PERIODIC SOLUTIONS FOR THE PROBLEM OF MOTION OF A HEAVY SOLID BODY AROUND A FIXED POINT

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The theory of generation of periodic solutions in canonic systems of near-integrable differential equations was developed by Poincaré for the purposes of celestial mechanics. In this paper we establish the applicability of these results to the classical problem of the motion of a heavy solid body with a fixed point. By the same token we have succeeded in essentially widening the class of periodic solutions appearing in this problem.

1. Perturbation of uniform rotations. The Hamiltonian function of the problem being analyzed has the form

$$F = F_0 + \mu F_1 \qquad (1.1)$$

Here F_0 is the kinetic energy, μF_1 is the system's potential energy (the chosen constant multiplier μ is the product of the body's weight by the distance from the center of gravity to the point of fixing). Canonic equations with Hamiltonian (1.1) have a cyclic integral, i.e. an area integral; by fixing it constant, we reduce the problem being examined to a system with two degrees of freedom, which we call reduced problem. When $\mu = 0$, we have the Euler-Poinsot case. In this unperturbed problem there exist particular isolated periodic solutions, namely, uniform rotations around the principal axes of the inertia ellipsoid. Let us ascertain whether the equations with Hamiltonian function (1.1) admit of periodic solutions if $\mu \neq 0$ but is very small.